BLIND JOINT ESTIMATION OF THE TECHNICAL PARAMETERS OF CONTINUOUS PHASE MODULATED SIGNALS.

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ABSTRACT

In this paper, a new non data aided estimator of the technical parameters of Continuous Phase Modulated (CPM) signals is proposed. It consists in estimating jointly the modulation index, the symbol period and the frequency offset. It is based on the following observations. Firstly, the inverse of the index is the smallest positive real number a CPM signal should be raised to in order to generate a deterministic harmonic signal; secondly, the frequencies of the harmonic signal are simply related to the symbol period and the carrier frequency. The asymptotic behavior of the estimation error is studied. If \( N \) is the number of signaling intervals, the estimate of the modulation index is shown to converge to a non Gaussian distribution at rate \( \frac{1}{\sqrt{N}} \), while the estimate of the frequency offset and the estimate of the symbol period converge at rate \( \frac{1}{\sqrt{NT^2}} \). Simulations results sustain our theoretical claims.

1. INTRODUCTION

Blind estimation of technical parameters characterizing the modulation used by a partially unknown transmitter is useful in certain civil or military applications. For instance, in the field of passive listening, we assume that the shaping filter is unknown. In this paper, a new non data aided estimator of the technical parameters of Continuous Phase Modulated (CPM) signals is proposed. It consists in estimating jointly the modulation index, the symbol period and the frequency offset. It is based on the following observation. A strictly positive real number \( g_0 = \frac{\hat{g}_0}{\delta f_0} \) is such that

\[
\psi(n_T) = \pi h \int_{-\infty}^{t} a_n g_0(u-nT_s) \, du = \pi h \sum_{n \in \mathbb{Z}} a_n \phi_s(u-nT_s)
\]

(\( a_n \) denotes the symbol sequence. It is assumed that \( a_n = \pm 1 \) for all \( n \), and that the sequence is centered and independent identically distributed. \( T_s \) is the unknown symbol period. Function \( g_0(t) \), classically called the shaping filter, is positive and non zero on the interval \([0, LT_s]\), where \( L \) is a positive integer. \( g_0(t) \) is normalized in such a way that \( \int_0^{LT_s} g_0(t) \, dt = 1 \). Therefore, function \( \phi_s(t) \) defined by

\[
\phi_s(t) = \begin{cases} 0 & \text{if } t < 0 \\ \int_0^t g_0(s) \, ds & \text{if } 0 \leq t \leq LT_s \end{cases}
\]

satisfies \( 0 \leq \phi_s(t) \leq 1 \) if \( 0 \leq t \leq LT_s \) and \( \phi_s(t) = 1 \) for \( t \geq LT_s \). Parameter \( h \) is called the modulation index and is therefore characterized by the fact that the phase variation induced by a symbol is equal to \( \pm \pi h \). We finally note that if \( nT_s \leq t \leq (n+1)T_s \), the phase \( \psi(n_T) \) of signal \( s_n(t) \) can be written as

\[
\psi(n_T) = \pi h \left[ \sum_{k=-\infty}^{n-L} a_k + \sum_{k=0}^{L-1} \phi_s(t-kT_s)a_{n-k} \right].
\]

Now consider that a CPM signal corrupted by an additive white Gaussian noise \( w_0(t) \) has been detected at the receiver side, and assume that the carrier frequency has been roughly compensated. The complex envelope \( y_0(t) \) of the received signal can be written as follows:

\[
y_0(t) = \alpha_0 s_0(t) e^{2\pi i f_0 t} + w_0(t).
\]

(4)

\( \alpha_0 \) represents an unknown complex gain, \( r \) is an unknown time delay and \( \delta f_0 \) denotes an unknown frequency offset.

In this paper, we address the non data aided joint estimation of the modulation index \( h \), the symbol period \( T_s \) and the frequency offset \( \delta f_0 \). As this contribution is motivated by applications to passive listening, we assume that the shaping filter is unknown.

We first mention that the use of the maximum likelihood criterion to estimate the unknown parameters is difficult to implement in case of CPM signals, as long as data symbols are unknown (see [5] for details). It is thus relevant to propose suboptimum estimators.

Blind estimation of the symbol period \( T_s \) can be achieved by using traditional cyclic methods, which consist in detecting the smallest positive cyclic frequency of a well chosen non-linear function of the received signal \( y_0(t) \). But these methods are known to be inappropriate when \( y_0(t) \) has a slight excess bandwidth. In our particular context, this means that cyclic methods are likely to fail when parameter \( L \) is large. On the other hand, a number of works have been devoted to the non data aided estimation of timing and frequency offset ([7], [8]) of CPM signals when every other parameters are assumed to be known. The problem of the estimation of the modulation index is comparatively less popular. Moreover, estimators proposed in [3] and [5] require the prior knowledge of the other parameters.

In section 2, we introduce the proposed joint estimator of \( \theta_0 = (h, T_s, \delta f_0) \). It is based on the following observation. A strictly positive real number \( g \) is such that

\[
\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T E(y_0(t)^g) e^{-2\pi i \hat{\alpha} t} \, dt \neq 0
\]

(5)

for at least a certain frequency \( \hat{\alpha} \) if and only if \( g \) is an integer multiple of \( g_0 = \frac{1}{\delta f_0} \). In other words, \( g_0 = \frac{1}{\delta f_0} \) is the smallest positive real number for which \( t \rightarrow E(y_0(t)^g) \) contains sinusoideal components. Furthermore, the frequencies \( \hat{\alpha} \) satisfying (5) for \( g = g_0 \) are given by \( \hat{\alpha} = \frac{\delta f_0}{g_0} + \frac{\pi h}{\delta f_0} \) for which \( k \) is an integer. This property is one of the result of the exhaustive study [2] of the cyclic properties of CPM signals. In section 4, we study the asymptotic performance of the proposed estimate \( \hat{\theta}_N = (\hat{h}_N, \hat{T}_N, \hat{\delta f}_N) \) of vector \( \theta_0 = (h, T_s, \delta f_0) \). We first study the noisless case and show that \( (\hat{h}_N - h), N^{3/2}(\hat{T}_N - T_s), N^{3/2}(\hat{\delta f}_N - \delta f_0) \) converges in distribution toward a distribution constructed from a 3-dimensional Brownian motion. The estimate \( \hat{h}_N \) of the modulation index thus converges at rate \( \frac{1}{\sqrt{N}} \). This is in contrast with the estimate proposed in [3] which converges at rate \( \frac{1}{\sqrt{NT^2}} \). The estimate \( \hat{T}_N \) of the symbol period converges at rate \( \frac{1}{\sqrt{NT^2}} \) and thus at the same rate as...
estimates based on cyclic methods (see [9]). Finally, the estimate \( \hat{\delta f}_N \) of the frequency offset \( \delta f_0 \) also converges at rate \( \frac{1}{\sqrt{N}} \). It is worth noting that the practical implementation of the estimate requires to raise the received signal to non integer powers. Thus, the phase of the received signal must be unwrapped thanks to a procedure which can produce some errors in the presence of additive noise. These phase unwrapping errors are quite difficult to take into account in the asymptotic analysis of the performance. While the asymptotic results remains exact in the noiseless case, the study of the noisy case requires to neglect the influence of phase unwrapping errors. In section 4, we compare the theoretical asymptotic distributions with the empirical ones. We study the values of the signal to noise ratio for which our theoretical results allow to predicate the behavior of the estimate.

2. THE PROPOSED ESTIMATE.

2.1. A cyclic property of CPM signals

The proposed estimate is based on a cyclic property of CPM signals which has been recently derived in [2].

**Proposition 1** Let \( r_s(t) \) be a CPM signal of modulation index \( f \). Then,

- if \( f \) is not an integer, then \( E(r_s(t)) = 0 \) for each \( t \);
- if \( f \) is a non zero even integer, function \( t \rightarrow E(r_s(t)) \) is periodic of period \( T_s \);
- if \( f \) is an odd integer, function \( t \rightarrow E(r_s(t)) \) is periodic of period \( 2T_s \).

Now consider a CPM signal \( s_a(t) \) whose phase \( \psi_a(t) \) is given by (3). We denote by \( y_a(t) \) the received signal given by (4). For the sake of simplicity, we consider the noiseless case and we assume that the complex gain \( a = \sum_{k=0}^{M-1} e^{j\pi k} \). In the sequel, we need to define signal \( y_a(t) \) for any positive real number \( g \). This requires some care because setting \( y_a(t)^\alpha = \left| y_a(t)^\alpha \right| \exp(i\delta f_0 t) \), one has to precise which particular determination of \( \text{Arg}(y_a(t)) \) is chosen. Recalling that \( y_a(t) = \exp(\delta f_0 t) \), we simply choose the determination

\[
\text{Arg}(y_a(t)) = \psi_a(t - \tau) + 2\pi \delta f_0 t,
\]

which can also be interpreted as the determination for which function \( t \rightarrow \text{Arg}(y_a(t)) \) is continuous. Using (6), the received signal \( y_a(t) \) raised to the power \( g \) can be written as:

\[
y_a(t)^g = s_a(t - \tau)^{g} e^{2\pi \alpha \delta f_0 t}.
\]

Using Proposition 1 and 2, we obtain immediately that \( E(r_N(\alpha, \beta)) \) converges to non zero values as \( N \rightarrow \infty \). As \( r_N(\alpha, \beta) \) converges almost surely towards \( E(r_N(\alpha, \beta)) \) for each \( (\alpha, \beta) \), this property suggests that the estimation of \( (g_0, \alpha_0) \) can be achieved by the maximization of \( E(r_N(\alpha, \beta))^2 \) w.r.t. \( (\alpha, \beta) \). Moreover, we notice that \( g_0, \alpha_0 \) and \( \beta_0 \) are related to the parameters of interest by: \( h = 1/g_0 \), \( T_s = T_s/(\alpha_0 - \beta_0) \) and \( \delta f_0 = (\alpha_0 + \beta_0)/(2g_0T_s) \). We thus define cost function \( J_N(g_0, \alpha, \beta) \) by

\[
J_N(g_0, \alpha, \beta) = |r_N(\alpha, \beta)|^2 + |r_N(\beta, \alpha)|^2
\]

and we propose to maximize \( J_N \) over a well chosen search domain. We put: \( (\alpha_N, \beta_N, \beta_N) = \arg \max_{(g_0, \alpha, \beta)} J_N(g_0, \alpha, \beta) \). The estimates of the technical parameters are finally given by \( h = 1/g_0 \), \( T_s = T_s/(\alpha_N - \beta_N) \) and \( \delta f_0 = (\alpha_N + \beta_N)/(2g_0T_s) \).
Remarks:

- At first glance, such a definition of search intervals may seem not so easy to do at the receiver side because the technical parameters are unknown. However, one may use an a priori information on \( \hat{\theta}_0 \).
- Indeed, it is reasonable to conjecture that an information on the possible values of parameter \( h \) is available and that the symbol period and the frequency offset have been roughly estimated beforehand: a basic estimation of \( T_c \) can indeed be achieved by detecting the bandwidth of the received signal \( y_\alpha(t) \), while the frequency offset \( \delta_0 \) can be roughly estimated by noticing that the average of the instantaneous frequency of \( y_\alpha(t) \) coincides with \( \delta_0 \). We also recall that the relevancy of the definition of search intervals \( I_\alpha \) and \( I_\beta \) rests on the assumption that \( |\gamma_0\delta_0| < \frac{1}{2} \). Since this assumption does not necessarily hold in practice, we propose firstly to estimate \( \delta_0 \) by using the average of the instantaneous frequency, secondly to compensate the frequency offset on the received signal and finally to refine the estimate of \( \delta_0 \) by using the present method.
- One could also replace \( \frac{\gamma_0(k)}{\gamma_0(k)} \) by \( y(k) \) in equation (8). However, the use of (8) provides better experimental results. This observation could be confirmed by the asymptotic analysis of the next section.

Due to the lack of space, we do not discuss this issue.

2.3. Practical implementation.

The practical implementation of the proposed estimate poses two problems.
- Function \( (g, (\alpha, \beta) \rightarrow \eta_\gamma(g, \alpha, \beta) \) is not convex. The maximization of \( \eta_\gamma \) cannot be achieved directly by using a gradient search algorithm: an exhaustive search is required at first. We thus propose the following procedure so as to maximize \( \eta_\gamma \). We first evaluate the values taken by the 1-dimensional cost function \( g \rightarrow \max_{\alpha, \beta} \eta_\gamma(g, \alpha, \beta) \) on a discrete grid of the search interval \( I_\alpha \). For a given \( g \), the above function can be evaluated thanks to the following steps.

1. Function \( \alpha \rightarrow \gamma_\gamma(g, \alpha) \) is evaluated on a discrete grid by an FFT algorithm.
2. \( |\gamma_\gamma(g, \alpha)|^2 \) is maximized w.r.t. \( \alpha \), where \( \alpha \) belongs to the discrete FFT grid and to the search interval \( I_\alpha \).
3. The point of the FFT grid which maximizes \( |\gamma_\gamma(g, \alpha)|^2 \) is used to initialize a Newton maximization algorithm, in order to refine the value \( \max_{\alpha} |\gamma_\gamma(g, \alpha)|^2 \).

The same approach is used in order to maximize \( \beta \rightarrow |\gamma_\gamma(g, \beta)|^2 \) w.r.t. \( \beta \in I_\beta \). Repeating these steps for each \( g \) leads to an initial estimate of \( (\gamma_\gamma(g_0, \alpha_0, \beta_0) \) which can be used as the initial point of a gradient search algorithm.

- The main problem of the present estimate comes from the computation of sequence \( (y(k))^n \) for \( n = 0, \ldots, N-1 \) for each non integer \( g \) of the search interval \( I_\alpha \). This crucial step requires indeed to unwrap the phase of \( g \) adequately. We first address the noiseless case, for which \( y(k) \) coincides with \( \alpha_0 \varphi \). In order to obtain a consistent estimate of \( g_0 \), one has to select for each \( k \) the determination defined by \( \text{Arg}(y(k)) = \text{Arg}(\gamma_0) + \psi(k) + 2\pi \Delta \varphi_0 k \). Now this can be achieved by noticing that the phase variation between two consecutive samples is so that \( \psi(k + 2\pi \Delta \varphi_0 k) - \psi(k - 1) + 2\pi \Delta \varphi_0 (k - 1) \) is large enough. Therefore, assuming that the correct determination \( \text{Arg}(y(k)) \) has been identified at time \( k = 1 \), \( \text{Arg}(y(k)) \) is defined as the determination for which \( |\text{Arg}(y(k)) - \text{Arg}(y(k - 1))| \) is minimum. In the noisy case, this unwrapping procedure is still used, but the influence of the additive noise \( w(k) \) may induce some phase unwrapping errors. Numerical results of section 4 illustrate the effect of such errors on the performance of the estimator.

3. ASYMPTOTIC ANALYSIS OF THE PROPOSED ESTIMATE.

We now study the asymptotic behavior of the estimate of the technical parameters \( \hat{\theta}_N = (\hat{h}_N, \hat{T}_N, \hat{\delta}_N) \). For the sake of simplicity, we assume that \( \exp(\pi i \bar{\theta} \sum_{n=-\infty}^{\infty} a_j) = 1 \) (using (3) and the fact that \( T_c = T_f \)), we obtain that for each \( n = 0 \ldots N - 1 \) and for each \( m = 0 \ldots M - 1 \), the phase of the discrete time signal \( s(nM + m) = s_n((nM + m)T_c - \tau) \) can be written as:

\[
\psi(nM + m) = \pi h \sum_{j=0}^{n-L} a_j + \sum_{j=0}^{L-1} a_{n-j} \phi_{j,m},
\]

where \( \phi_{j,m} \) is defined by \( \phi_{j,m} = \phi_0(T_c + mT_c/M + T_c - \tau) \) and we assume without restriction that \( 0 \leq T_c - \tau \leq T_c \).

3.1. The noiseless case.

We first consider the noiseless case, i.e. \( g(k) = \alpha_0 \varphi \) for each \( k = 0 \ldots N - 1 \). As constant \( \alpha_0 \) does not play any role in the following, we simply set \( \alpha_0 \) from now on in order to simplify the notations.

Before presenting the main results of this section, we first study the behavior of \( r_N(g_0, \alpha_0) \) in order to get some insights on the parameters which may influence the performance of the estimates. Recalling that \( a_0 = \frac{1}{\sqrt{N}} + \hat{g}_0 \Delta \varphi_0, r_N(g_0, \alpha_0) \) can be written as:

\[
r_N(g_0, \alpha_0) = \frac{1}{N M} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} (-1)^{n} \exp i g_0 \psi(nM + m + \pi \varphi) \exp \left\{ \pi \frac{\pi}{2} \right\}
\]

(11)

We also recall that \( \exp(i g \psi(nM + m + \pi \varphi)) \) coincides with \( \pm(-1)^n \) and we assume that it is equal to \((-1)^n \). Using (10) and the fact that \( g_0 = 1 \), \( \exp(i g \psi(nM + m)) \) can be written as:

\[
\exp(i g \psi(nM + m + \pi \varphi)) = (-1)^n \exp i \pi \sum_{j=0}^{L-1} a_{n-j} \phi_{j,m}
\]

(12)

Expanding the left-hand side of (12), we easily obtain that:

\[
(-1)^n \exp i g \psi(nM + m + \pi \varphi) = \left( \sum_{j=0}^{L-1} \cos(\pi \phi_{j,m}) \right) + \varepsilon(n)\psi(nM + m) + \cdots
\]

Using (13), it can easily be shown that \( r_N(g_0, \alpha_0) \) converges toward \( \alpha_0 \) as \( N \rightarrow \infty \). In other words, \( |\alpha_0|^2 \) can be interpreted as the asymptotic magnitude of the peaks of cost function \( J_N \). The above result gives us already the insight that the modulus of \( \alpha \) has a crucial influence on the performance of the proposed estimate. In the sequel, we define \( \p = |\alpha| \) and \( \varphi = \text{Arg}(\lambda) \). We also need to define \( \hat{\varepsilon}(n) = \frac{1}{N} \sum_{m=0}^{M-1} \varepsilon(nM + m) \) as the deterministic constant defined by \( \hat{\varepsilon}(n) = \frac{1}{N} \sum_{m=0}^{M-1} \cos(\pi \phi_{j,m}) \exp(-i \pi \varphi) \).

Using (13), it can be easily shown that \( r_N(g_0, \alpha_0) \) converges toward \( \alpha_0 \) as \( N \rightarrow \infty \). In other words, \( |\alpha_0|^2 \) can be interpreted as the asymptotic magnitude of the peaks of cost function \( J_N \). The above result gives us already the insight that the modulus of \( \alpha \) has a crucial influence on the performance of the proposed estimate. In the sequel, we define \( \p = |\alpha| \) and \( \varphi = \text{Arg}(\lambda) \). We also need to define \( \hat{\varepsilon}(n) = \frac{1}{N} \sum_{m=0}^{M-1} \varepsilon(nM + m) \) as the deterministic constant defined by \( \hat{\varepsilon}(n) = \frac{1}{N} \sum_{m=0}^{M-1} \cos(\pi \phi_{j,m}) \exp(-i \pi \varphi) \).

In order to study the asymptotic behavior of the estimation errors \( \hat{g}_N - g_0, \hat{T}_N - T_c, \) and \( \hat{\delta}_N - \delta_0 \), it is quite useful to remark that the estimates can also be defined as \( (\hat{g}_N, \hat{T}_N, \hat{\delta}_N) = \arg \max_{(g, T, \delta)} J_N(g, T, \delta) \), where cost function \( J_N(g, T, \delta) \) is equal to:

\[
\hat{J}_N(g, T, \delta) = J_N(g, T, \delta) + \frac{2T_c}{2T} + \frac{2T_c}{2T} + g \delta,
\]

(14)
In particular, we note that \( J_N(g_0, T_c, \delta f_0) = J_N(g_0, \alpha_0, \beta_0) \). The usual approach consists in expanding up to the second order the cost function \( J_N \) around \((g_0, T_c, \delta f_0)\) and then to conjecture that the 3-dimensional estimation error has the same asymptotic behavior as the product \(-H_{N}^{-1}\nabla J_N\), where \( H_{N} \) and \( \nabla J_N \) respectively represent the Hessian matrix and the gradient of \( J_N \) at point \((g_0, T_c, \delta f_0)\). It is thus sufficient to study separately the behaviors of the derivatives of function \( J_N \) at the point \((g_0, T_c, \delta f_0)\). After some algebra, one can show that each of the latter derivatives can be written as a function of the discrete time processes \( \tau_1(n) = \text{Im}(\hat{\epsilon}_1(n)e^{-i\varphi} + \hat{\epsilon}_2(n)e^{i\varphi}) \) and \( \alpha_n \). In order to illustrate this claim, we mention that the first derivative \( \partial J_N(g_0, T_c, \delta f_0) \) of \( J_N \) w.r.t. \( g \) at point \((g_0, T_c, \delta f_0)\) is equal to:

\[
2\pi h\rho \left( \mu + \sum_{n=0}^{N-1} \left( \frac{\sum_{j=0}^{n-L} a_j}{N^{3/2}} \right) (\sum_{n=0}^{N-1} \tau_1(n))^{-1} \right) + \mathcal{O}(\frac{1}{N^{1/2}})
\]

where \( \mu \) is the deterministic term defined by:

\[
\mu = -\frac{2}{M} \sum_{m=0}^{N-1} \sum_{j=0}^{L-1} \phi_{j,m} \sin \pi \phi_{j,m} \prod_{k=0}^{L-1} \cos \pi \phi_{k,m} \cos \left( \frac{\pi m}{M} + \varphi \right).
\]

Notation \( \mathcal{O}(\cdot) \) stands for bounded in probability.

The asymptotic behavior of the estimates can be characterized by using the so-called functional central limit theorem. In order to introduce this result, we need to define on \([0,1]\) the following stochastic processes:

\[
W_1^{(N)}(t) = \frac{1}{N} \sum_{n=0}^{N-1} a_{n-L}
\]

\[
W_2^{(N)}(t) = \frac{1}{N} \sum_{n=0}^{N-1} \tau_1(n)
\]

\[
W_3^{(N)}(t) = \frac{1}{N} \sum_{n=0}^{N-1} \tau_2(n)
\]

where \([x]\) denotes the greatest integer less than or equal to \( x \).

We also define the 3-dimensional stochastic process \( W^{(N)}(t) = (W_1^{(N)}(t), W_2^{(N)}(t), W_3^{(N)}(t)) \). The functional central limit theorem states that process \( W^{(N)}(t) \) converges in distribution toward a 3-dimensional Brownian motion \( W(t) = (W_1(t), W_2(t), W_3(t)) \) of covariance matrix \( \Gamma \), where:

\[
\Gamma = \sum_{k \in \mathbb{Z}} E \left( \begin{pmatrix} a_{n+k-L} & \tau_1(n+k) & \tau_2(n+k) \\ \tau_1(n+k) & a_{n-L} & \tau_2(n) \\ \tau_2(n+k) & \tau_2(n) & a_{n-L} \end{pmatrix} \right).
\]

We recall that a Brownian motion of covariance matrix \( \Gamma \) is a zero mean Gaussian process such that \( E(W(t)W(s)^T) = \Gamma \min(t,s) \). In particular, this implies that if \( F \) is a functional defined on a well chosen space of functions defined on \([0,1]\), then \( F(W^{(N)}) \) converges in distribution toward the random variable \( F(W) \). In order to illustrate this claim, we mention for example that \( \int_0^1 W_1^{(N)}(t)dt = \frac{2\pi h\rho}{\mathcal{N}} \sum_{n=0}^{N-1} \sum_{j=0}^{L-1} a_j \) converges in distribution toward \( \int_0^1 W_1(t)dt \). Using this kind of ideas, one can show the following theorem:

\[\text{Theorem 1}\]

Denote by \( \xi \) and \( \chi \) the random variables defined by

\[
\xi = \mu + \int_0^1 W_1(t)dt W_2(1) - \int_0^1 W_1(t)dt W_3(t) + 3\int_0^1 (1 - 2t) W_1(t)dt (\int_0^1 (1 - 2t) W_2(t))
\]

\[
\chi = \int_0^1 W_1(t)dt W_2(t) - (\int_0^1 W_1(t)dt)^2 - 3\int_0^1 (1 - 2t) W_1(t)dt W_3(t)
\]

Then \( (\tilde{N} h_N - h), N^{3/2} (T_N - T_c), N^{3/2} (\delta f_N - \delta f_0) \) converges in distribution toward the following random vector:

\[
\left( \begin{pmatrix} -\frac{h_N}{\tau \pi} \frac{\xi}{\tau} \int_0^1 (1 - 2t) dW_2(t) \\ -\frac{3h_N}{2\tau \pi} \int_0^1 (1 - 2t) dW_2(t) \right) + \mathcal{O}(\frac{1}{N^{1/2}})
\]

Comments:

- The estimate \( h_N \) of \( h \) converges at rate \( \frac{1}{\sqrt{n}} \). This is in contrast with the estimate proposed in [3] for which the convergence rate is equal to \( \frac{1}{N} \). The estimate \( T_N \) converges toward \( T_c \) at rate \( \frac{1}{\sqrt{n}} \) and thus at the same rate as in case of traditional cyclic methods ([9]). Finally, the estimate \( \delta f_N \) of \( \delta f_0 \) also converges at rate \( \frac{1}{\sqrt{n}} \). Note also that the 3-dimensional asymptotic mean square error does not depend on the value of the frequency offset \( \delta f_0 \).

- The 3-dimensional estimation error is asymptotically proportional to \( \frac{1}{\sqrt{n}} \). As expected, the performance crucially depends on the value of \( \rho \) (and thus on the shaping filter \( g_0(t) \) used at the transmitter side).

- Random variable \( \int_0^1 (1 - 2t) W_2(t) \) is a zero-mean Gaussian variable. This implies that the estimate \( T_N \) of \( T_c \) is asymptotically unbiased and that the distribution \( N^{3/2} (T_N - T_c) \) tends to a Gaussian distribution. On the other hand, the asymptotic behaviors of the estimates \( h_N \) and \( \delta f_N \) are rather unconventional.

- Using basic symmetric properties of the probability measure of Brownian motions, it can be shown that the estimate \( \delta f_N \) of \( \delta f_0 \) is also asymptotically unbiased. As we shall see in section 4, the estimate \( h_N \) of the modulation index is biased. The theoretical analysis of this bias requires more involved arguments and is not addressed in this paper due to the lack of space.

3.2. The noisy case

We still assume without restriction that constant \( \alpha_0 = 1 \) so that \( y(k) = e^{i\epsilon(k)+2\eta\Delta f_0} + w(k) = e^{i\epsilon(k)} (1 + \bar{w}(k)) \)

where \( \bar{w}(k) = e^{-i\epsilon(k)+2\Delta f_0}(1 - \bar{w}(k)) \) as \( w(k) \) is complex Gaussian and independent of \( e^{i\epsilon(k)} \). It is easily seen that \( \bar{w}(k) \) is still complex Gaussian and independent of \( e^{i\epsilon(k)} \).

We put \( \bar{w}(k) = \rho(k)e^{i\delta(k)} \), where \( \delta(k) \in [-\pi, \pi] \). \( \rho(k) \) is Rice distributed, while the probability distribution of \( \delta(k) \) has an even (well known) probability density. As mentioned in subsection 2.3, the calculation of \( y(k) \) requires to unwrap the phase of \( y(k) \). The additive noise may produce some phase unwrapping errors which are unfortunately very difficult to take into account. A rigorous extension of the above asymptotic analysis seems therefore difficult. However, Theorem 1 may be extended if phase unwrapping errors are neglected, i.e. it is assumed that the unwrapped phase of \( y(k) \) coincides with the sequence \( \psi(k) + 2\pi \Delta f_0 k + \delta(k) \) for each \( k \). Using the same approach as in the previous subsection, we obtain after some algebra that the vector \( (\tilde{N} h_N - h), N^{3/2} (T_N - T_c), N^{3/2} (\delta f_N - \delta f_0) \) converges in distribution toward the following random vector:

\[
\left( \begin{pmatrix} -\frac{h_N}{\tau \pi} \frac{\xi}{\tau} \int_0^1 (1 - 2t) dW_2(t) \\ -\frac{3h_N}{2\tau \pi} \int_0^1 (1 - 2t) dW_2(t) \right) + \mathcal{O}(\frac{1}{N^{1/2}})
\]

1Notation \( \text{Im} \) stands for imaginary part of
tributions of the estimation errors modulated signals (i.e. the shaping filter is the raised-cosine of or-
given by
\[ \xi_{\nu} = \mu - \rho E(\delta(k) \sin(\phi(k))) + \left( \int_0^1 W_1(t) dt \right) W_{2,\nu}(1) - \int_0^1 W_1(t) dW_{2,\nu}(t) + 3 \left( \int_0^1 (1 - 2t) W_1(t) dt \right) \int_0^1 (1 - 2t) dW_{2,\nu}(t). \]

Random process \( W_{2,\nu}(t) = (W_1(t), W_{2,\nu}(t), W_{3,\nu}(t)) \) represents a 3-dimensional Brownian motion. Due to the lack of space, we just mention that its covariance matrix \( \Gamma_{\nu} \) depend on the signal-to-noise ratio. More details will be given in an extended version of the present paper.

4. SIMULATIONS AND RESULTS

In this section, we compare theoretical predictions to empirical results. We first give the parameters of the simulations. The number \( N \) of signaling intervals is set to \( N = 1000 \). The time delay \( \tau \) is equal to \( \tau = 0.21 T_s \). The additive noise is assumed to be white in the frequency interval \([- \frac{1}{T_s}, \frac{1}{T_s}]\), so that its variance \( \sigma^2 \) is given by \( \sigma^2 = \frac{\pi^2}{2} N_0 \). Results presented in the sequel are obtained by using either 1REC modulated signals (i.e. the shaping filter \( g_0(t) \) is given by \( g_0(t) = 1 \) on \([0, T] \) and \( g_0(t) = 0 \) elsewhere) or 3RC modulated signals (i.e. the shaping filter is the raised-cosine of order \( L = 3 \), given by \( g_0(t) = \frac{1}{\pi T_s} (1 - \cos(\frac{2\pi}{T_s} t)) \)). The modulation index \( h \) is equal to \( h = 0.7 \). The unknown frequency offset is set to \( \delta_0 = 0 \) without restriction. Finally, the oversampling factor \( \mathcal{M} \) is equal to \( \mathcal{M} = 4 \). In order to evaluate function \( J_N \) and thus to estimate the technical parameters, we use the practical implementation described in subsection 2.3. More precisely, function \( g = \max_{\alpha, \beta} J_N(g, \alpha, \beta) \) is evaluated on a grid whose step is equal to \( 10^{-5} \). For each point of this grid, we use 4 iterations of a Newton algorithm to calculate the values of \( \alpha \) and \( \beta \) that maximize \( J_N(g, \alpha, \beta) \). Finally, 200 iterations of a gradient search algorithm are used to refine the estimates.

The following figures represent the empirical and theoretical distributions of the estimation errors \( N(h_N - h)/h \) and \( N^{3/2}(\delta f_N - \delta f_0) \) in different simulation contexts. Empirical distributions are given by normalized histograms which are based on 2000 realizations of random variables \( h_N \) and \( \delta f_N \). Theoretical distributions correspond to the dotted line. Figures 1 and 2 represent the distribution errors in case of 3RC signals respectively for \( E_b/N_0 = 25dB \) and \( E_b/N_0 = 15dB \). We first observe that the histograms fit to the theoretical limit distribution for both estimates. The performance obtained either for \( E_b/N_0 = 25dB \) and \( E_b/N_0 = 15dB \) is almost the same : in case of 3RC signals, the performance is quite insensitive to the noise level as long as phase unwrapping errors do not occur. These simulation results also confirm that the estimate \( \delta f_N \) is unbiased and that the estimate \( h_N \) is biased. Figure 3 shows that the performance of the joint estimation is better in case of 1REC signals than in case of 3RC signals.

Using other simulation results, we observe that when \( E_b/N_0 \) is less than 12dB, the approximations formulated in order to derive the limit distribution are no longer valid because of a significant number of phase unwrapping errors. In this case, theoretical predictions do not fit to the empirical results.

Furthermore, simulations have shown that a large number \( N \) of observed signaling intervals is required so that the empirical distribution of \( N^{3/2}(T_N - T_0) \) fits to theoretical results. Indeed, \( N^{3/2}(T_N - T_0) \) coincides with the sum of a random variable which has the same distribution as \( \frac{2\pi}{T_s} \int_0^1 (1 - 2t) dW_{3,\nu}(t) \) and of second term which converges toward zero at rate \( \frac{1}{\sqrt{N}} \). Intuitively, the empirical distribution of the estimation error shall thus coincide with the theoretical distribution only if this second term can be neglected w.r.t. the random variable \( \frac{2\pi}{T_s} \int_0^1 (1 - 2t) dW_{3,\nu}(t) \). Now we observed that this random variable may have a very slight variance (less than \( 10^{-15} \) in all our simulation contexts). Therefore, it is clear that the empirical distribution of \( T_N \) shall not fit to the theoretical results for \( N = 1000 \). Nevertheless, the asymptotic study of section 3 can be extended so as to predict the behavior of \( T_N \) in the most general case. This issue will be addressed in further works.

5. REFERENCES